A Closed Form Solution of Minimum Line Segment Distance via Constrained Optimization

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1 Introduction

When modelling a continuous structure within a discrete system such as a computer simulation, a Piecewise Linear model is often substituted for ease of implementation. A common example of this practice comes from quadratic and cubic splines, such as Hermite and Catmull-Rom. Presented here is a (hopefully) straightforward derivation of the closed form line segment distance originally discussed briefly in [1]. This is similar in construction and equivalent to [2].

2 Derivation

For distance between line segments, first define a line segment L_i in parameter t between points P_i and P_{i+1},

$$L_i(t) = P_i + (P_{i+1} - P_i) \cdot t.$$

Defining the direction vector $\vec{d_i} = P_{i+1} - P_i$, we have

$$L_i(t) = P_i + \vec{d_i} \cdot t.$$

The distance squared between any two line segments L_i, in parameter t, and L_j, in parameter u, is given by $f(t,u) = |L_i(t)L_j(u)|^2$. Since the function $f(t,u)$ is quadratic, we are guaranteed a global minimum at $\partial f \equiv \vec{0}$. This can be obtained via a constrained optimization over the unit square, as seen in Figure 1.

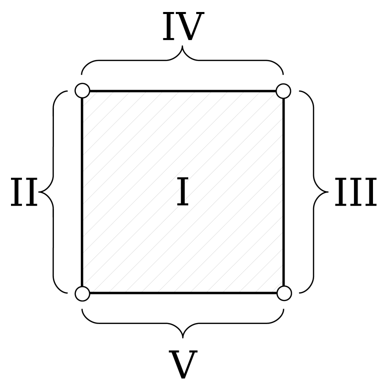

Figure 1:

For case `I`, optimizing over $(0,1) \times (0,1)$, consider the system

$$\begin{cases} \frac{\partial f}{\partial t} = 0 \\ \frac{\partial f}{\partial u} = 0 \end{cases}$$

Defining $\alpha_i = |\vec{d_i}|^2$, $\beta = \vec{d_i} \cdot \vec{d_j}$, and $\gamma_i = \vec{d_i} \cdot (P_i - P_j)$, this is

$$\begin{cases} \frac{\partial f}{\partial t} = \alpha_i \cdot t - \beta \cdot u + \gamma_i = 0 \\ \frac{\partial f}{\partial u} = \alpha_j \cdot u - \beta \cdot t - \gamma_j = 0 \end{cases}$$

After Gaussian Elimination, we arrive at

$$t = \frac{\gamma_j \cdot \beta - \alpha_j \cdot \gamma_i}{\beta^2 - \alpha_i \cdot \alpha_j},\ u = \frac{\gamma_i \cdot \beta - \alpha_i \cdot \gamma_j}{\beta^2 - \alpha_i \cdot \alpha_j} \quad \texttt{(I)}$$

For cases `II-V`, we optimize over $\{0\} \times (0,1)$, $\{1\} \times (0,1)$, and the other two cases symmetric about the parameter. Thus, we have the optimal parameter values of

$$
\begin{array}{lll}
t = \frac{\partial f}{\partial t}|_{u=0} = -\frac{\gamma_i}{\alpha_i}, & u = 0 & (\texttt{II}), \\
t = \frac{\partial f}{\partial t}|_{u=1} = \frac{\beta-\gamma_i}{\alpha_i}, & u = 1 & (\texttt{III}), \\
u = \frac{\partial f}{\partial u}|_{t=0} = \frac{\gamma_j}{\alpha_j}, & t = 0 & (\texttt{IV}), \\
u = \frac{\partial f}{\partial u}|_{t=1} = \frac{\beta+\gamma_j}{\alpha_j}, & t = 1 & (\texttt{V}).
\end{array}
$$

So far, we have optimized over open intervals containing the interior and each edge of the unit square. When one of these formulae satisfy $0 \leq t, u \leq 1$, the shortest distance lies between two non-endpoints or between one endpoint and one non-endpoint. If no case satisfies the the unit interval condition, the shortest distance must lie between two endpoints. This dissolves to finding the minimum of four point distances.

Acknowledgements

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References

[1] Lipari, N. and Borst, C., “Evaluation of Stereoscopy and Lit Shading for a Counting Task in Knot Visualization”, The 2007 International Conference on Computer Graphics and Virtual Reality.

[2] Hoffmann, G., “Distance between Line Segments”, `http://www.fho-emden.de/ \ ~hoffmann/xsegdist03072004.pdf`, July 11, 2005, date accessed: Apri 25, 2007.